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1993 J. Phys. A: Math. Gen. 26 4273

(http://iopscience.iop.org/0305-4470/26/17/032)

J. Phys. A: Math. Gen. 26 (1993) 4273–4279. Printed in the UK

# Scaling and multifractality in a river network model with flow-dependent meandering

Takashi Nagatani

College of Engineering, Shizuoka University, Hamamatsu 432, Japan

Received 22 September 1992, in final form 27 April 1993

**Abstract.** A river network model with a variable exponent of drainage basin is presented. The river network model is an extended version of the Scheidegger river model to take into account a flow-dependent meandering. The scaling behaviour of river-size distribution, the scaling of drainage basin and the multifractal structure of flow (channel discharge) distribution are investigated by computer simulation. It is shown that the exponent of drainage basin changes continuously from 1.5 (the value of the Scheidegger river) to 1 (the value of a linear river) with increasing exponent $\gamma$ of flow-dependent meandering, and the exponent of cumulative river-size distribution changes from 0.33 to 0.0. It is found that the partition function

$$Z(q) \equiv \sum_i I_i^q$$

of flow distribution scales as $Z(q) \approx L^{\zeta(q)}$ where $I_i$ is the flow (channel discharge) of water passing over the site $i$ within the river network and the summation ranges over all sites. The multiscaling exponent $\zeta(q)$ of flow distribution changes with the exponent $\gamma$ of flow-dependent meandering. The dependence of $f$–$\alpha$ spectra on values of $\gamma$ are discussed.

## 1. Introduction

Recently, there has been increasing interest in fractal growth phenomena such as diffusion-limited aggregation (DLA), cluster–cluster aggregation, rough surfaces and river networks [1–9]. Branched river networks are among nature's most common patterns, spontaneously producing fractal structure [2, 9]. Rivers have been studied extensively by a wide variety of researchers with a variety of techniques and goals. Some models have been constructed for the evolution of an entire drainage network [10–12]. Scheidegger's river model is the simplest model which reveals the essential features of river formation. The cumulative size distribution of rivers in Scheidegger's model satisfies the power law

$$P(\geqslant S) \approx S^{-1/3} \tag{1}$$

and the area $A$ of the drainage basin scales as

$$A \approx L^{1.5} \tag{2}$$

where $L$ is the length of the mainstream in the downstream direction [13].

Very recently, in Scheidegger's river model, it was found that the flow distribution shows a typical multifractal structure [14]. It was shown that the partition function

$$Z(q) \equiv \sum_i I_i^q$$

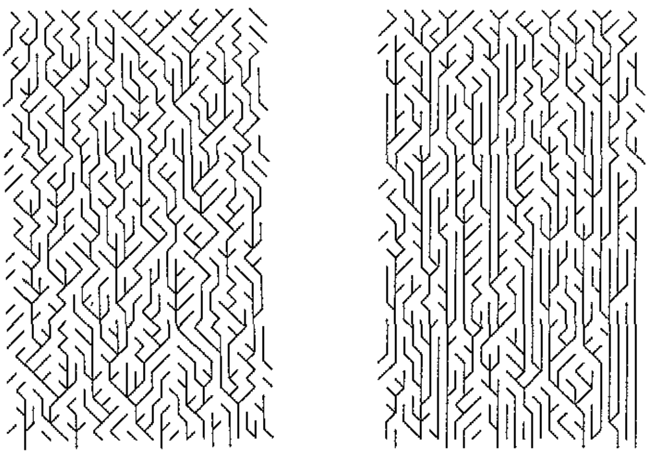
of flow distribution scales as $Z(q) \approx L^{\zeta(q)}$ where $I_i$ is the flow (channel discharge) of water passing over the bond $i$ within the river network. It was also found that the river-width distribution shows multifractality if the width $w$ of a river scales as $w \approx I^\beta$.

In the field of potamology the best-known empirical law is that known as Hack's law. This asserts that the area $A$ of the drainage basin scales as $A^{1/2} \approx L^{1/1.2}$ and hence the fractal dimension of the mainstream can be seen to be 1.2. For all rivers, the fractal dimension of the mainstream calculated individually by the coarse-graining method falls in the range 1.1 to 1.3, with a mean value of 1.2 [9]. However, a river network model with a variable fractal dimension (or exponent of the drainage basin) has not been proposed until now.

A river network model with a variable exponent of the drainage basin is presented. The river network model is an extended version of Scheidegger's river model to take into account flow-dependent meandering. The size distribution of rivers, the scaling of the drainage basin and the multifractal property of the flow distribution are investigated by use of Monte Carlo simulation. The exponent $\tau - 1$ of cumulative river-size distribution is found to change continuously from 1/3 to 0 with increasing exponent $\gamma$ of flow-dependent meandering, the exponent $d_b$ of the drainage basin changes simultaneously from 1.5 to 1. The effect of flow-dependent meandering on the multifractality of the flow distribution is shown.

## 2. The new model

First the river network model with flow-dependent meandering is introduced. Scheidegger's river model is extended to take into account flow-dependent meandering. In Scheidegger's river model, a river meanders right or left at each site with probability 1. In our model, a river meanders right or left with probability $p$ less than 1 and goes straight on with probability $1-p$. The meandering of a river is assumed to depend on the flow (channel discharge) of the river. The river tends to go straight with increasing flow. In order to take into account the straight flow of a river, we consider the extended river network model on a triangular lattice rather than the square lattice of Scheidegger's river model. The flow of water on the triangular lattice goes not only right down or left down as Scheidegger's model, but also straight down. Rains are assumed to fall steadily and uniformly on the sites of the oblique triangular lattice (figure 1). One unit of water is injected into each site per unit time, then, the fallen raindrops 'walk' down the slope. When two or three raindrops collide with each other, they join and make one larger drop, which runs down just as before the collision. The flow is not permitted to split. Flows are allowed to go straight down with flow-dependent probability $1-p=1-I^{-\gamma}(\gamma>0)$, and to go right down or left down with probability $p$. As a result, rivers go down in a preferred direction (downstream) and do not contain any loops. All branches are directed upstream. The flow (channel discharge) $I_i$ at site $i$ is defined as the amount of flowing water through site $i$ per unit time. The flow at site $i$ is proportional to the area of its drainage basin connecting upstream to site $i$. Each site of the river network can be characterized by the flow of



(a) (b)

**Figure 1.** Typical river network patterns generated by the river network model with flow-dependent meandering. These runs were done on a $15 \times 50$ triangular lattice under a periodic lateral boundary condition for illustration. The river pattern obtained for (a) $\gamma=0.5$; (b) $\gamma=1.0$.

water. If the site is labelled by the position $(m, n)$, the flow (channel discharge) satisfies the equation

$$\begin{aligned} I(m+1, n) = {} & T(m-1, n; I(m-1, n)) I(m-1, n) \\ & + [1-T(m, n-1; I(m, n-1))] w(m, n-1) I(m, n-1) \\ & + [1-T(m, n+1; I(m, n+1))][1-w(m, n+1)] I(m, n+1) + 1 \end{aligned} \tag{3}$$

where $m$ indicates the downstream direction, $T(m, n)$ denotes the realization of the flow direction at site $(m, n)$, which is equal to 1 when the flow at site $(m, n)$ goes straight down and 0 when the flow goes right down or left down, $T(m, n; I(m, n))$ is given by

$$T(m, n; I(m, n)) = \begin{cases} 1 & \text{probability } 1 - I(m, n)^{-\gamma} \\ 0 & \text{probability } I(m, n)^{-\gamma} \end{cases} \tag{4}$$

$w(m, n)$ denotes the realization of the flow direction at site $(m, n)$, which is equal to 1 when the flow at the site $(m, n)$ goes right down and 0 when the flow goes left down, and $w(m, n)$ is given by

$$w(m, n) = \begin{cases} 1 & \text{probability } 1/2 \\ 0 & \text{probability } 1/2. \end{cases} \tag{5}$$

In the limit $\gamma=0$, the river network model of equation (3) reproduces Scheidegger's river model [13].

We perform computer simulations of equation (3) for the triangular lattice $1000 \times 1000$. The flow $I(m, n)$ on each site is calculated under a periodic lateral boundary condition. The river at site $(m, n)$ goes straight down with probability $1 - I(m, n)^{-\gamma}$ and goes right or left down with probability $I(m, n)^{-\gamma}$. For illustration, figure 1 shows typical patterns obtained by a small-size simulation. The patterns (a) and (b) are obtained, respectively, at $\gamma=0.5$ and $\gamma=1.0$ for size $15\times 50$. Rivers tend to go straight down with increasing exponent $\gamma$.

To calculate the size distribution of rivers (or channel discharge) we define the cumulative river-size distribution $P(\geqslant S)$ as

$$P(\geqslant S)\equiv\int_S^\infty p(S')\,\mathrm{d}S'$$

where $p(S)$ indicates the distribution with size $S$ and $S$ is the are of the drainage basin. The area $S$ of the drainage basin equals the channel discharge. Figure 2 shows the log–log plot of cumulative river-size distributions for $\gamma=0.0$, 0.3, 0.5 and 1.0. The cumulative size distribution scales as

$$P(\geqslant S)\approx S^{-(\tau-1)} \tag{6}$$

The values of the exponent $(\tau-1)$ are given, respectively, by $(\tau-1)=0.33, 0.26, 0.18$ and $0.04\ \pm0.02$ for $\gamma=0.0$, 0.3, 0.5 and 1.0. The value of $\gamma=0.0$ agrees with that of Scheidegger's river model. The exponent $(\tau-1)$ approaches zero for $\gamma$ larger than 1.0. *The exponent $(\tau-1)$ of the cumulative river-size distribution changes continuously* from 0.33 (the value in Scheidegger's model) to 0.0 (the value for a linear river) with increasing $\gamma$. Table 1 shows the values of the exponents $(\tau-1)$ of the cumulative river-size distribution with the exponents of the drainage basin. Flow-dependent meandering has an important effect on cumulative river-size distribution.

In order to further characterize the river network, it is necessary to derive the multifractal structure of flow distribution. We study the scaling behaviour of the partition function

$$Z(q)\equiv\sum_i I_i^q$$

of the moments of flow. Figure 3 shows the log–log plot of the moments $Z(q)$ against

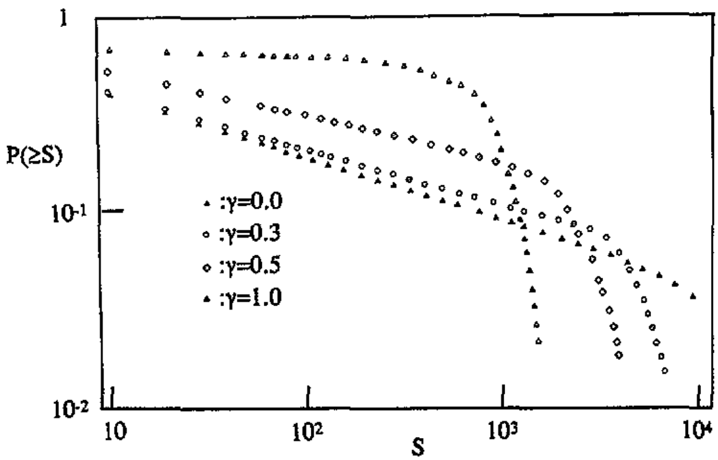



**Figure 2.** Log–log plots of cumulative river-size distribution $P(\geqslant S)$ against size $S$ (the area of the drainage basin) for $\gamma=0.0$, 0.3, 0.5 and 1.0.

**Table 1.** The exponent $(\tau-1)$ of the cumulative river-size distribution, the exponent $d_b$ of the drainage basin and the minimum value $\alpha(\infty)$ of $f-\alpha$ spectrum for the exponent $\gamma$ of flow-dependent meandering. The errors in the exponents $(\tau-1)$ and $d_b$ are about $\pm 0.02$. The errors in $\alpha(\infty)$ are about $\pm 0.03$.

| | | | | |
|---|---|---|---|---|
| $\gamma$ | 0.0 | 0.3 | 0.5 | 1.0 |
| $(\tau-1)$ | 0.33 | 0.26 | 0.18 | 0.04 |
| $d_b$ | 1.50 | 1.27 | 1.17 | 0.99 |
| $\alpha(\infty)$ | 0.51 | 0.68 | 0.79 | 0.93 |

the length $L$ of the system in the downstream direction for $\gamma=0.3$. It is confirmed that the partition function scales for large $L$ as

$$Z(q)\approx L^{\zeta(q)} \tag{7}$$

Figure 4 indicates the log–log plot of the third moment $Z(3)$ against $L$ for various values of $\gamma$. Figure 5 shows the multiscaling exponents $\zeta(q)$ against $q$ for $\gamma=0.0$, 0.3, 0.5 and 1.0. The curve of $\gamma=0.0$ corresponds to that of the original Scheidegger river model. The flow distribution shows a typical multiscaling character. For different values of $\gamma$, the multiscaling exponent $\zeta(q)$ changes to definitely different structures. The multiscaling exponent $\zeta(q)$ depends strongly on the value of $\gamma$ or flow-dependent meandering.

The exponent $\zeta(0)$ gives the scaling exponent of the number of total bonds covering the river network. $\zeta(0)$ equals 1 for any $\gamma$. We note that the exponent $\zeta(0)$ is not 2 but 1 since the partition function $Z(q)$ is defined for a constant value (1000) of system size. The exponent $\zeta(1)$ gives the scaling exponent of the cumulative number of bonds covering the river network. $\zeta(1)$ equals 2 for any $\gamma$.

The partition function $Z(1)$ equals the integrated value of rainfall throughout the area since the amount of the channel discharge at a fixed position $L$ is proportional to the amount of rainfall throughout the upstream area. Therefore $Z(1)\approx L^2$ and $\zeta(1)=2$. For a sufficiently large $q$, $\zeta(q)/q$ gives the scaling exponent of largest flow. The exponent $\zeta(q)/q(q\to\infty)$ is also consistent with the exponent of the drainage basin. In table 1, the exponent $d_b$ of the drainage basin obtained from $d_b=$

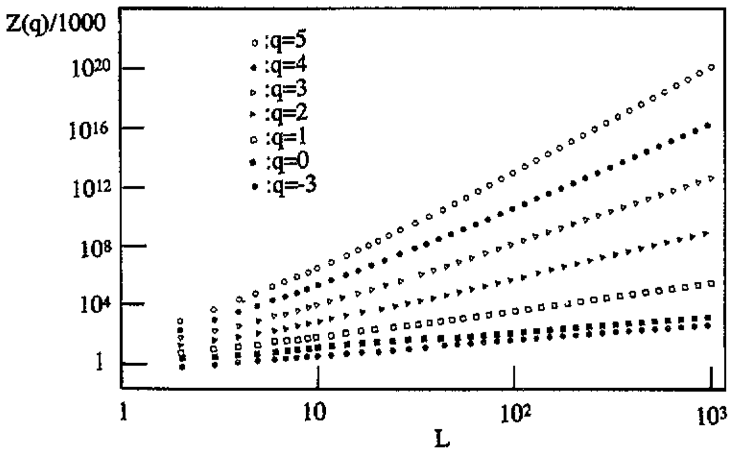



**Figure 3.** Log–log plot of the moments $Z(q)$ against the system size $L$ for $\gamma=0.3$.

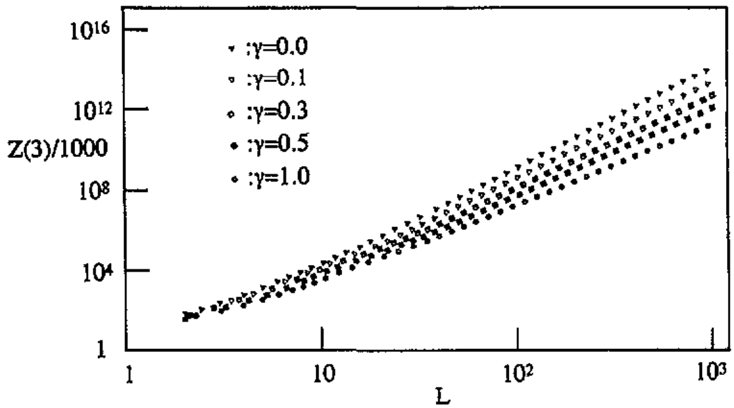



**Figure 4.** Log–log plot of the third moment $Z(3)$ against the system size $L$ for various values of $\gamma$.

$\zeta(q)/q(q\to\infty)$ is shown for various values of $\gamma$. For $\gamma=0.0$, $\zeta(q)/q(q\to\infty)$ gives the value $d_b=1.50\pm0.02$ of Scheidegger's river model. In the limit $\gamma\to\infty$, the exponent $d_b$ equals 1 since the river becomes linear. The exponent $d_b$ is larger than 1. In table 1, the value $d_b=0.99\pm0.03$ for $\gamma=1.0$ is consistent with $d_b=1$ within the numerical accuracy.

The multifractal spectrum seems to behave very differently for $q\leqslant0$ from that for larger $q$s. However, there seems to be no phase transition in the multifractal spectrum at $q=0$ since the minimum value of the channel discharge is 1 and the minimum normalized flow does not decrease faster than the power law. In order to characterize the multifractality of the flow distribution, it is convenient to normalize the flow. The normalized flow $I'_i$ at the site $i$ is defined as $I'_i=I_i/Z(1)$. The partition function $Z'(q)$ of the normalized flow is given by $Z'(q)=Z(q)/\{Z(1)\}^q$. For a sufficiently large $L$, the partition function scales as $Z'(q)\approx L^{-\tau(q)}$. With the Legendre transformation of $\tau(q)$, we obtain the $f-\alpha$ spectrum $f(q)=q\alpha(q)-\tau(q)$ where $\alpha(q)=\partial\tau(q)/\partial q$ is the variable conjugate to $q$. The $f-\alpha$ spectrum of $\gamma=0.0$ corresponds to that of Scheidegger's river model. The maximum value $f(0)$ of $f(\alpha)$ gives the scaling exponent $\zeta(0)=1$ for

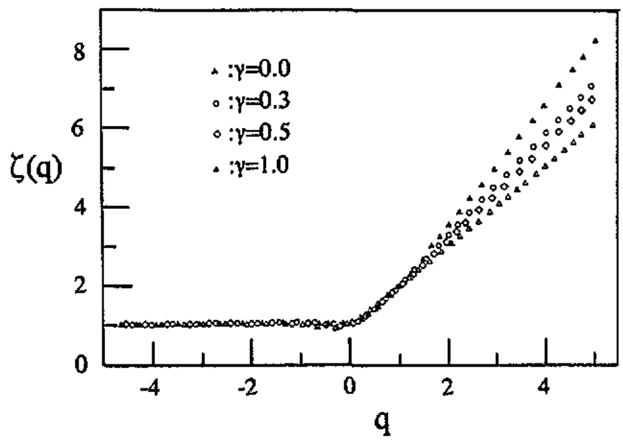



**Figure 5.** Behaviour of the multiscaling exponent $\zeta(q)$ for $\gamma=0.0$, 0.3, 0.5 and 1.0. The case $\gamma=0.0$ corresponds to that of Scheidegger's river model.

any $\gamma$. The minimum value of $\alpha$ gives the maximum fraction of flow. The minimum value $\alpha(\infty)$ is exactly related to the exponent $d_b$ of the drainage basin:

$$\begin{aligned}\alpha(\infty) &= \left(\frac{\partial\tau}{\partial q}\right)_{q=\infty} \\ &= -\left[\frac{\partial}{\partial q}\left\{\frac{\ln Z(q)}{\ln L}\right\}\right]_{q=\infty} + \frac{\ln Z(1)}{\ln L} \\ &= 2 - d_b \end{aligned} \tag{8}$$

where $\ln Z(1)/\ln L = 2$. The minimum values $\alpha(\infty)$ obtained from the simulation are respectively, given by 0.51, 0.68, 0.79 and $0.93 \pm 0.03$ for $\gamma = 0.0$, 0.3, 0.5 and 1.0. Table 1 shows the values $\alpha(\infty)$ with $d_b$. The relationship (8) is satisfied within the accuracy of the simulation.

## 3. Summary

A river network model with flow-dependent meandering has been presented. The scaling behaviour of river-size distribution and the multifractal property of flow distribution has been investigated using computer simulation. It was found that the exponent $d_b$ of the drainage basin changes continuously from 1.5 (the value of Scheidegger's river) to 1 (the value of a linear river) with increasing exponent $\gamma$ of flow-dependent meandering, and the exponent of cumulative river-size distribution changes from 0.33 to 0.0. The multifractality of flow distribution was derived, and the dependence of the $f-\alpha$ spectrum on flow-dependent meandering was demonstrated.

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